

Option Pricing Theory and Its Applications

by

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1. Introduction

Option pricing theory has a long and illustrious history. The first scientific study of options was done by Louis Bachelier in 1900 in a remarkable work that also contained several pioneering results in the theory of stochastic processes. Unfortunately, Bachelier's work fell into obscurity and remained undiscovered by economists for over fifty years. Little research was done on options during that time, but subsequently the field became more active. Key contributions were made in the 1960's by Samuelson [1965] and Samuelson and Merton [1969]. It was not until the 1970's, however, that a completely satisfactory theory of option pricing was developed. This new theory, which revolutionized the field, was initiated by Black and Scholes [1973] and extended in important ways by Merton [1973]. Their path-breaking articles have formed the basis for nearly all subsequent work in the area.

Recent work on option pricing has tended to fall into two broad categories. The first category is foundational work whose objective is to refine, extend, and elucidate the basic ideas underlying the Black-Scholes theory. In Section 2 of our review, we present this line of research by rederiving the basic Black-Scholes results from the perspective of subsequent work. In the second category are articles which have developed applications of this basic theory. It is the tremendous variety and power of these applications that has given the field much of its vitality. As one illustration of this, we show in Section 3 how option pricing theory can be fruitfully applied to a classical issue in financial economics, the intertemporal portfolio problem. In Section 4, we briefly survey a number of other applications of the theory.

2. The martingale approach to option pricing

In this section, we will develop option pricing theory from a somewhat different perspective than the way in which it was originally presented in the articles of Black and Scholes [1973] and Merton [1973]. We will exploit the martingale connection of an arbitrage free price system as first observed by Cox and Ross [1976] and formalized by Harrison and Kreps [1979]. This approach will lead naturally to a discussion in Section 3 on some fundamental issues in portfolio theory.

To simplify notation, we will take the Black-Scholes model of securities prices as a starting point and later discuss directions for generalization. Throughout we will consider an economy with a long horizon. However, we focus our attention only on the time interval $[0, 1]$.

2.1. The setup

Let there be two securities, one risky and another riskless. The risky security does not pay dividends on $[0, 1]$ and has a price process:

$$S(t) = \exp \left\{ \left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W(t) \right\}, \quad (2.1)$$

where μ and σ are two strictly positive constants, and where $\{W(t)\}$ is a standard Brownian motion under a probability P .¹ The interpretation of μ and σ will be clear if we take the differential of (2.1):

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t).$$

The σ is the *instantaneous* standard deviation and μ the *instantaneous* expected value of the rate of return of the risky asset. The riskless asset does not pay dividends on $[0, 1]$ and has a price process

$$B(t) = \exp\{rt\} \quad t \in [0, 1]. \quad (2.2)$$

The vector price processes (B, S) will be termed a *price system*.

An investor in the economy is interested in trading in the two securities to achieve an optimal random wealth at time one. We shall assume that the random time one wealths in which he is interested have finite second moments² and that he has no sources of income other than his endowed wealth at time zero.

One immediate question that comes to mind is how do we know the price system written down is a *reasonable* price system? In the least, the price system should not allow us to create something out of nothing or create *free lunches*. In order to formulate what we mean by *creating*, we have to talk about what types of investment strategies, or *trading strategies*, are allowed and what kind of information can be used by investors. For a rich model of securities trading, we would certainly like to include as many trading strategies as possible, as long as there is no free lunch that can arise.

The information an investor has at any time t is the past realization of the price system. Since the riskless asset has a purely deterministic price process, at time zero, an investor knows its future price behavior completely. Thus, the information that an investor has learned between any time t and time zero is just the realizations of the risky asset price process there. We shall use \mathcal{F}_t to denote the information that an investor possesses at time t .³ Compactly denoted, the information structure of an investor is $\mathbf{F} = \{\mathcal{F}_t; t \in [0, 1]\}$, or a *filtration generated by S* . Note that from (2.1) we can write

$$W(t) = \frac{\ln S(t) - (\mu - \frac{1}{2}\sigma^2)t}{\sigma} \quad t \in [0, 1].$$

Thus knowing the realizations of S is equivalent to knowing those of the Brownian motion W . Thus the filtration (information) generated by S is equivalent to the filtration (information) generated by W .

A process is said to be consistent with the information structure if its values at time t can only depend upon the information at that time. Alternatively, a consistent process is also said

¹A standard Brownian motion is a Brownian motion that starts at zero at time zero.

²Any random variable that has a finite second moment must have a finite mean and a finite variance.

³Mathematicians call \mathcal{F}_t the sigma-field generated by $\{S(s); 0 \leq s \leq t\}$.

to be *adapted*. A process is said to be *predictable* if its values at time t depend only upon the information strictly before time t . A predictable process is certainly adapted.

A trading strategy is a pair of predictable processes $\{\alpha(t), \theta(t)\}$. For every $t \in (0, 1]$, interpret $\alpha(t)$ and $\theta(t)$ to be the number of shares of the riskless and risky security, respectively, held at time t before time t trading. That is, $\alpha(t)$ is the number of shares of the riskless security carried by an investor from an instant before time t into t , likewise for $\theta(t)$. Then naturally, the informational constraint is that the values of $(\alpha(t), \theta(t))$ be dependent only upon the information strictly before time t . As for $(\alpha(0), \theta(0))$, they represent the initial holdings of the two securities and are therefore nonstochastic.

A trading strategy is said to be *simple* if it is bounded and if it only changes its values or trades at finitely many of points of time. The number of those time points, although finite in number, can be arbitrarily large.

Since a simple strategy can easily be implemented in real life, it seems that we should at least allow all the simple strategies that satisfy an appropriate budget constraint. Formally, a simple strategy (α, θ) , with trading dates $0 = t_0 < t_1 < \dots < t_n = 1$, is *admissible* if

$$\alpha(t_i)B(t_i) + \theta(t_i)S(t_i) = \alpha(t_{i+1})B(t_i) + \theta(t_{i+1})S(t_i) \quad \forall i = 0, 1, \dots, n-1. \quad (2.3)$$

The left-hand-side of (2.3) is the value of the strategy at time t_i before the trading; and the right-hand-side is the value after the trading at t_i . That they are equal is a natural budget constraint, since an investor does not have income other than his endowed time zero wealth. The value of the strategy at time zero will be equal to an investor's time zero wealth. Here we note that any simple strategy generates a time one wealth with a finite second moment. It can be easily verified that (2.3) implies that

$$\alpha(t)B(t) + \theta(t)S(t) = \alpha(0)B(0) + \theta(0)S(0) + \int_0^t \alpha(s)dB(s) + \int_0^t \theta(s)S(s) \quad t \in [0, 1]. \quad (2.4)$$

That is, the value of a strategy at time t is equal to its initial value plus accumulated capital gains or losses. Any trading strategy, not necessarily simple, that satisfies (2.4) will henceforth be termed a *self-financing* strategy for obvious reasons. (Implicit in this statement is the requirement that all the integrals are well defined.)

Let H denote the space of admissible strategies. We will specify H carefully later; for the time being, we only say that it includes all the admissible simple strategies and any element of it must be self-financing. Here we are obliged to explain why we are not content with simple strategies, since it seems that those are the ones that are likely to be carried out in real life. Besides certain technical reasons, here we can only point out that some financial assets, like options, cannot be *manufactured* by simple strategies. Thus, for the *richness* of the model, we would like to also

consider strategies that are not simple. For technically inclined readers, we note that the space of simple strategies lacks the closure property that is convenient for many purposes.

Now we shall formalize what we mean by a *free lunch*. In words first, a free lunch is a sequence of admissible trading strategies whose initial costs go to a nonpositive real number and whose time one value goes to a nonzero positive random variable. Formally, a free lunch is a sequence of admissible trading strategies $(\alpha^n, \theta^n) \in H$, $n = 1, 2, \dots$, such that $\lim_n \alpha^n(0)B(0) + \theta^n(0)S(0) \leq 0$ and that $\lim_n \alpha^n(1)B(1) + \theta^n(1)S(1) \geq k$, where k is some nonzero positive random variable.⁴

Now we are ready to discuss an important consequence of the no free lunch condition. We shall first fix some notations. Putting $S^*(t) = S(t)/B(t)$ and $B^*(t) = 1$, (B^*, S^*) is a *normalized* or *discounted* price system, where the riskless asset is the numeraire.

The following theorem, an application of Harrison and Kreps [1979] and Kreps [1981] gives the connection between the normalized price system and martingales.

Theorem 2.1. *Suppose that (B, S) admits no free lunches. Then there exists a probability Q under which S^* is a martingale. The probabilities Q and P are equivalent in the sense that an event is of probability zero under P if and only if it is of probability zero under Q . Moreover, the derivative of Q with respect to P , denoted by dQ/dP , has a finite variance. Conversely, if there exists such a probability Q then there are no free lunches for simple admissible strategies.*

Remark 2.1. In a finite state model with finitely many discrete periods, the above theorem follows from the absence of *simple free lunches*, as an application of the Farkas Lemma (see, e.g., Holmes [1975], p.92). A simple free lunch is an admissible strategy whose initial cost is nonpositive and whose time one value is a nonzero positive random variable. Moreover, when there is an agent whose preferences have a von Neumann-Morgenstern expected utility representation, dQ/dP can be viewed as his marginal utility per unit of probability P . When there are an infinite number of states, the absence of simple free lunches is no longer sufficient for the above theorem for some technical reasons. We refer interested readers to Kreps [1981] for a host of related issues.

Normally we would not expect price processes to be martingales. A set of sufficient conditions for that to be true is: investors are risk neutral and the interest rate is zero. In the above theorem, the normalization takes out the positive interest rate and the change of probability subsumes risk aversion. We shall call the probability Q a *martingale measure*. We will see shortly that a martingale measure is just a Cox and Ross [1976] risk neutral probability.

The task now is to verify that there exists a martingale measure for the price system (B, S) . Our approach will be through construction. We will define a probability and show that it is a martingale measure.

⁴The convergence of the time zero investment is in real numbers. The convergence of time one value is in the sense of the expectation of the square of the difference.

Put

$$\xi = \exp \left\{ - \left(\frac{\mu - r}{\sigma} \right) W(1) - \frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2 \right\}. \quad (2.5)$$

Since $W(1)$ is a normally distributed random variable with mean zero and variance 1, ξ is a log-normal random variable. A log-normal random variable has finite arbitrary moments and is strictly positive. Define a probability Q by

$$Q(A) = E[1_A \xi] \quad \forall A \in \mathcal{F}_1.$$

where A is any distinguishable event at time one and 1_A is an indicator function of the event A taking the value 1 if the true state of the nature lies in A and taking the value 0 otherwise. The density of Q with respect to P is just ξ . The two probabilities P and Q are equivalent since ξ is strictly positive.

Now we claim that Q is a martingale measure. To prove this, we need a mathematical result, the Girsanov Theorem, which states that the Brownian motion $\{W(t)\}$ under P becomes a Brownian motion plus a drift term under Q (see, e.g., Liptser and Shiriyayev [1977]).

Theorem 2.2. (*Girsanov*) *Putting*

$$z^*(t) = W(t) + \left(\frac{\mu - r}{\sigma} \right) t,$$

then $\{z^*(t)\}$ is a Standard Brownian motion under Q .

The drift term in the above theorem comes from equation (2.5). In fact, the Girsanov Theorem holds more generally than is stated in the above theorem. When the three parameters μ , σ , and r are not constant but are stochastic processes themselves, we simply replace $\left(\frac{\mu - r}{\sigma} \right) t$ by the integral of $\left(\frac{\mu - r}{\sigma} \right)$ from 0 to t . When there are more than one risky securities and μ and σ are a vector and a matrix, respectively, we replace $\frac{\mu - r}{\sigma}$ by $\sigma^{-1}(\mu - r)$. (Of course, in the general case, there are some additional regularity conditions.)

Given the Girsanov Theorem, we can write for all $0 \leq s \leq t \leq 1$:

$$\begin{aligned} S^*(t) &= \exp \left\{ \left(\mu - \frac{1}{2} \sigma^2 - r \right) t + \sigma W(t) \right\} \\ &= S^*(s) \exp \left\{ - \frac{1}{2} \sigma^2 (t - s) + \sigma (z^*(t) - z^*(s)) \right\}. \end{aligned} \quad (2.6)$$

Given $S^*(s)$, $S^*(t)$ is equal to $S^*(s)$ times a log-normal random variable with unit mean under Q . Letting $E^*[\cdot | \mathcal{F}_t]$ be the conditional expectation under Q given the information at time t , we immediately have

$$E^*[S^*(t) | \mathcal{F}_s] = S^*(s).$$

which simply says that $\{S^*(t)\}$ is a martingale under Q . Thus according to Theorem 2.1, (B, S) admits no free lunches for admissible simple strategies.

The question that remains is whether the existence of a martingale measure implies no free lunch for a strategy space containing non-simple strategies. For example, consider all the self-financing strategies. The answer is no by an example due to Harrison and Kreps [1979]. They conceive of a self-financing strategy termed the *doubling strategy*. This strategy is like borrowing to double your bet in roulette each time you lose. As long as you can borrow an unbounded amount and can bet infinitely often, you are sure to win in the end. By way of doing this, you will essentially create something out of nothing. This kind of scheme is certainly self-financing. It is feasible in our economy since there are infinitely many trading opportunities in $[0, 1]$ and there are no bounds on the numbers of shares of securities that an investor can hold, and hence no limit on the amount of borrowing that can be done.

There are two approaches that one can take to make (B, S) a viable model of securities prices, while allowing non-simple strategies. The first one is suggested by Harrison and Kreps. To implement a doubling strategy, one has to allow the possibility that one's wealth can go negative and be unbounded from below before one actually wins. Hence prohibiting an investor from having negative wealth will certainly rule out the doubling strategies. Harrison and Kreps conjectured that the nonnegative wealth constraint may also rule out all the free lunches. Dybvig [1980] confirmed their conjecture.

The second approach is to put a certain condition on the number of shares of securities that an investor can hold. The idea is that a doubling strategy will involve shorting more and more of the riskless security and going long in the risky one in some state of the nature. If we put a bound on the position in the risky security that one can take over time, then the doubling strategy will not be implementable. Note that a bound need only be put on the position of the risky security, since the same bound will affect the position of the riskless security through the budget constraint. It turns out that we can admit strategies more general than the bounded ones. They only need to satisfy a square-integrability condition:

$$E^* \left[\int_0^1 (\theta(t) S^*(t))^2 dt \right] < \infty, \quad (2.7)$$

where $E^*[\cdot]$ is the expectation at time zero under Q . This approach has been adopted by Harrison and Pliska [1981] and Duffie and Huang [1985]. It turns out that any self-financing strategy satisfying (2.7) is the limit of a sequence of admissible simple strategies.⁵ In words, a non-simple self-financing strategy satisfying (2.7) can be approximated arbitrarily closely by an admissible

⁵ That is, if θ satisfies (2.7), then there is a sequence of simple θ_n , $n = 1, 2, \dots$, such that $E^* \left[\int_0^1 ((\theta(t) - \theta^n(t)) S^*(t))^2 dt \right] \rightarrow 0$ as $n \rightarrow \infty$.

simple strategy that trades very very frequently. It can be easily shown that any simple strategy satisfies (2.7).

The two approaches turn out to be functionally equivalent for a class of utility maximization problems. The nonnegative wealth constraint rules out free lunches, but still allows suicidal strategies—strategies that are mirror images of free lunches and throw money away. The condition (2.7), however, rules out both free lunches and suicidal strategies. For nonsatiated investors, suicidal strategies will never be employed. Thus the two approaches are equivalent for a nonsatiated investor in that they yield the same solution set for the investor. Here we take the second route by defining H to be the collection of self-financing strategies (α, θ) satisfying (2.7) such that $\alpha(1)B(1) + \theta(1)S(1) \in L^2(P)$, where we have used $L^2(P)$ to denote the space of random variables consistent with \mathcal{F}_1 and having finite variances under P . Thus defined, H is a linear space and that makes our discussion to follow on dynamic spanning easier. (If we took the first approach, the space of admissible trading strategies will not be a linear space.)

Now we summarize in the following theorem:

Theorem 2.3. *(B, S) admits no free lunches with respect to H .*

Before we leave this section, we note two things. First, if we substitute for P the martingale measure Q , we will only change the drift term of the price process of the risky security. The instantaneous standard deviation term will not be affected. To see this, we apply Itô's lemma to (2.6) and get

$$dS^*(t) = \sigma S^*(t)dz^*(t). \quad (2.8)$$

Under Q , S^* is a martingale and does not have a drift term. The instantaneous standard deviation under Q of the rate of return on S^* is still σ . This property is not particular to the normalized price process. For S , we have

$$dS(t) = rS(t)dt + \sigma S(t)dz^*(t). \quad (2.9)$$

Under the martingale measure, the instantaneous expected rate of return of S is equal to the riskless rate and the σ is unaffected. So, as we mentioned previously, the martingale measure is just the risk neutral probability of Cox and Ross [1976].

Second, Itô's lemma implies that $(\alpha, \theta) \in H$ if and only if

$$\alpha(t) + \theta(t)S^*(t) = \alpha(0) + \theta(0)S^*(0) + \int_0^t \theta(s)\sigma S^*(s)dz^*(s). \quad (2.10)$$

Now note a result in the theory of stochastic integration: Let θ satisfy (2.7). Then

$$\int_0^t \theta(s)\sigma S^*(s)dz^*(s) \quad t \in [0, 1]$$

is a martingale having finite second moment under Q . It follows that the left-hand-side of (2.10), the value of the strategy $(\alpha, \theta) \in H$ in units of the riskless security, is a martingale with a finite second moment under Q :

$$\alpha(t) + \theta(t)S^*(t) = E^* [\alpha(1) + \theta(1)S^*(1) | \mathcal{F}_t] \quad t \in [0, 1]. \quad (2.11)$$

A consequence of this is that if we know the final value of a strategy $(\alpha, \theta) \in H$ and θ , then we know α by rearranging (2.11):

$$\alpha(t) = E^* [\alpha(1) + \theta(1)S^*(1) | \mathcal{F}_t] - \theta(t)S^*(t). \quad (2.12)$$

That is, the final wealth generated by an admissible trading strategy is *completely* determined by the number of shares of the risky security that is held over time. The number of shares of the riskless security held over time will then be determined through the budget constraint as manifested by (2.12).

2.2. Dynamic spanning and the martingale representation theorem

Recall that an investor's task is to find an optimal time one wealth through trading dynamically in the two securities. We have learned from the previous section that there do not exist free lunches for admissible strategies. In this section, we will first examine the kinds of time one random wealth that can be achieved by trading in the two long-lived securities. We will show that although there are only two long-lived securities, any time one wealth that has a finite second moment under both P and Q is achievable through some strategy in H . Through the sequential trading opportunity, one can turn the two long-lived securities into an achievable final wealth space infinite in dimension.

In particular, a European call option written on the risky asset expiring at time one with any exercise price $K > 0$ is achievable and thus has a well-defined price. We will compute its price and demonstrate the strategy that manufactures it.

To fix notations, let $L^2(Q)$ be the collection of random variables that are consistent with the information at time one and have finite second moments under Q . We will first show that any element of $L^2(P) \cap L^2(Q)$ is achievable through some strategy in H . This is a direct consequence of the *martingale representation theorem*, due originally to Kunita and Watanabe [1967]. Before recording this theorem, we note that since $z^*(t) = W(t) + (\frac{\mu - r}{\sigma})t$, the information structure \mathbf{F} is also generated by z^* .

Theorem 2.4. (*Kunita and Watanabe*) Suppose that the information structure \mathbf{F} is generated by z^* , a Brownian motion under Q . Then any finite second moment martingale $\{m(t)\}$ under Q consistent with \mathbf{F} can be represented as a stochastic integral with respect to $\{z^*(t)\}$:

$$m(t) = m(0) + \int_0^t \eta(s) dz^*(s) \quad t \in [0, 1],$$

where $\{\eta(t)\}$ is predictable and satisfies

$$E^* \left[\int_0^1 |\eta(t)|^2 dt \right] < \infty. \quad (2.13)$$

(Note that although here $\{z^*(t)\}$ is a one dimensional Brownian motion under Q , the above theorem applies also to the case where the Brownian motion is multi-dimensional.)

Now let $x \in L^2(P) \cap L^2(Q)$. We want to show that there exists $(\alpha, \theta) \in H$ that manufactures x . Since x has a finite second moment under Q , xe^{-r} does too. Hence, $\{E^*[xe^{-r}|\mathcal{F}_t]\}$ is a finite second moment martingale under Q . By the martingale representation theorem, we know there exists $\{\eta(t)\}$ satisfying (2.13) such that

$$\begin{aligned} xe^{-r} &= E^*[xe^{-r}] + \int_0^1 \eta(t) dz^*(t) \\ &= E^*[xe^{-r}] + \int_0^1 \frac{\eta(t)}{S^*(t)\sigma} S^*(t)\sigma dz^*(t). \end{aligned} \quad (2.14)$$

Putting $\theta(t) \equiv \eta(t)/(S^*(t)\sigma)$ and defining $\alpha(t)$ by (2.12), we can easily verify that $(\alpha, \theta) \in H$ and it manufactures x .⁶ Hence any element x of $L^2(P) \cap L^2(Q)$ is achievable through some strategy (α, θ) in H . From the previous section, we also know how to compute the price for x : the price for x at time t is $E^*[xe^{-(1-t)r}|\mathcal{F}_t]$.

Consider now a European call option written on the risky security with an exercise price of $K > 0$ and an expiration date of time one. The payoff of this option at time one is $y \equiv \max[S(1) - K, 0]$. Note that $S(1)$ is a log-normal random variable both under P and Q , so $y \in L^2(P) \cap L^2(Q)$. That is, this call option can be manufactured by some $(\alpha, \theta) \in H$. Its price at time t , denoted by $C(t)$, is

$$C(t) = E^* \left[ye^{-(1-t)r} | \mathcal{F}_t \right]. \quad (2.15)$$

Carrying out the computation, we get the Black-Scholes formula.

From the Black-Scholes formula, we know that the price of a European call option at any time t depends upon $S(t)$, σ , r , K , and t . Since σ , r , and K are constants, we can write $C(t) = C(S(t), t)$. It is easily verified that $C(S(t), t)$ is twice continuously differentiable with respect to $S(t)$ and once continuously differentiable with respect to t . Itô's lemma implies

$$\begin{aligned} C(S(t), t)e^{-rt} &= C(0) + \int_0^t C_S(S(s), s) S^*(s)\sigma dz^*(s) \\ &\quad + \int_0^t e^{-rs} \left(\frac{1}{2} \sigma^2 S^2(s) C_{SS}(S(s), s) + rS(s) C_S(S(s), s) + C_\tau(S(s), s) - rC(S(s), s) \right) ds. \end{aligned} \quad (2.16)$$

⁶ A technical argument can show that (α, θ) is predictable.

One can verify that

$$E^* \left[\int_0^1 (C_S(S(t), t) S^*(t))^2 dt \right] < \infty.$$

Thus the second term on the right-hand-side of (2.16) is a martingale under Q . Recall that the normalized price for any achievable time one wealth is a martingale under Q . Hence the third term on the right side of (2.16) is also a martingale. A martingale does not have any drift term. It then follows that the third term on the right side of (2.16) must be zero and therefore

$$\frac{1}{2} \sigma^2 S^2(t) C_{SS}(S(t), t) + r S(t) C_S(S(t), t) + C_t(S(t), t) - r C(S(t), t) = 0. \quad (2.17)$$

Since $S(t)$ is log-normally distributed and therefore has a support equal to the positive real line excluding zero, the option price must be a solution to the partial differential equation:

$$\frac{1}{2} \sigma^2 x^2 C_{xx}(x, t) + r x C_x(x, t) + C_t(x, t) - r C(x, t) = 0 \quad \forall x \in (0, \infty) \quad t \in [0, 1]. \quad (2.18)$$

This is the so called *fundamental partial differential equation for valuation* in the option pricing literature. Here we derived this partial differential equation as a by-product of the option pricing formula.

Given (2.16), it is then clear what is the strategy that manufactures the call option: $\theta(t) = C_S(S(t), t)$ and $\alpha(t)$ is defined by (2.12). The number of shares of the risky security held at time t is just the partial derivative of the option price with respect to $S(t)$. The number of shares of the riskless security held is then determined through the budget constraint. One can also verify easily that Itô's lemma can be applied to C_S . Thus C_S is itself an Itô process, which fluctuates very fast and is not a simple strategy on S .

We note that in the above demonstration of the strategy for a call option, the only property that we made use of $C(S(t), t)$ is that it has certain continuous partial derivatives for Itô's lemma to work. Thus we have in fact demonstrated a general method in constructing a strategy for an achievable time one wealth whose value at any time t is a function of $S(t)$ and t , to which the Itô's lemma can be applied. We will see this point again in the next section.

Using the martingale approach in option pricing theory has several advantages. First, the interplay between the existence of free lunches and admissible trading strategies is clearly revealed. Second, it brings forth the observation that a *reasonable* price system should, in the least, be a martingale after a normalization and a change of probability. Third, the space of reachable final wealths can be explicitly characterized. Not only options, but also everything else that lies in $L^2(P) \cap L^2(Q)$ is reachable. Hence we have the amazing fact that the number of traded securities is finite in number, but the space of reachable time one wealths is of infinite dimension. This is not a mystery, however, since we have an uncountably infinite number of trading opportunities.

2.3. Some Generalizations

We have mentioned that the martingale approach can be applied to very general stochastic environments. The following is such a scenario: There are a finite number of securities indexed by $j = 0, 1, 2, \dots, J$, which, for simplicity, do not pay dividends. Except for the 0-th security, all the others are risky. Denote by $\{B(t), S(t) = (S_1(t), \dots, S_J(t))^T\}$ the price processes for the riskless and the risky securities, where T denotes the transpose. We assume that $B(t) = \exp\{\int_0^t r(s)ds\}$ and that S is an J -dimensional Itô process:

$$S(t) = S(0) + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW(s) \quad \forall t \in [0, 1],$$

where W is a J -dimensional standard Brownian motion. Itô's lemma implies that $S^*(t) \equiv S(t)/B(t)$ can be written as

$$\begin{aligned} S^*(t) &= S^*(0) + \int_0^t \frac{1}{B(s)}(\mu(s) - r(s)S(s))ds + \int_0^t \frac{\sigma(s)}{B(s)}dW(s) \\ &\equiv S^*(0) + \int_0^t \mu^*(s)ds + \int_0^t \sigma^*(s)dW(s). \end{aligned}$$

Now under some regularity conditions, and especially with the nonsingularity of the $J \times J$ matrix process $\{\sigma(t)\}$, there exists a unique martingale measure. Therefore, as an application of Theorem 2.1, there are no free lunches for self-financing simple strategies. When we consider the space of self-financing trading strategies $\{\alpha(t), \theta(t) = (\theta_1(t), \dots, \theta_J(t))^T\}$ that satisfy the following square-integrability condition:

$$E^*\left[\int_0^1 \theta(t)\sigma^*(t)\sigma^*(t)^T\theta(t)^T dt\right] < \infty,$$

no free lunches exist, where as usual $E^*[\cdot]$ denotes the expectation under the martingale measure Q . Under some regularity conditions and a generalization of Theorem 2.4, any final wealth in $L^2(P) \cap L^2(Q)$ is reachable by a self-financing strategy satisfying the above integrability condition. The value at time t of any reachable final wealth, in units of the 0-th asset, is just the conditional expectation at that time of the final wealth under the martingale measure. Readers are referred to Cox and Huang [1986a] and Harrison and Kreps [1979] for complete details.

We have also noted earlier that, using the martingale approach to price a call option in the Black-Scholes context, the price of an option can be computed by evaluating the conditional expectation of (2.15). As a consequence, the fundamental partial differential equation for valuation (2.18) becomes an implication of the Black-Scholes option pricing formula; it is no longer the vehicle through which the call option price is solved, as it was in the original treatments of Black and Scholes [1973] and Merton [1973]. The two approaches are indeed equivalent in the simple set up of the Black-Scholes economy.

In many situations, however, it is impossible to evaluate the relevant conditional expectation explicitly. For example, consider valuing an American put option expiring at time one in the set up

of Section 2.1. It is well known that it may be optimal for the holder of such an option to exercise it before the expiration date (see Merton [1973]). Then the value of the put option depends upon the optimal exercise policy and will be given by the supremum of the conditional expectation over all exercise strategies. In this case, no analytic expression has yet been found for the value of the option. However, we do know that the fundamental partial differential equation for valuation will be satisfied by the put option price before exercise and that the exercise strategy will be chosen to maximize the value of the option. Hence one can use numerical techniques for solving partial differential equations to find the approximate optimal exercise policy and the approximate value of the put option. Readers may consult Brennan and Schwartz [1977a] and Parkinson [1977] for details.

Even without the optimal exercise problem, conditional expectations will usually be impossible to evaluate explicitly in situations with general stochastic environments such as the one depicted in the beginning of this section. Hence it will again be necessary to solve the fundamental partial differential equation numerically in order to obtain specific results.

It is important to note that all of the results we have described express the value of an option in terms of the value of some other security or set of securities. It is the derivation of relative pricing results using only a few properties of the underlying equilibrium that characterizes option pricing as a separate field of study. In any situation in which the relative pricing methodology cannot be applied, option valuation must be considered in the context of a complete general equilibrium model (see, e.g., Rubinstein [1976]). In those cases, option pricing theory becomes indistinguishable from the more general theory of asset valuation.

3. Existence and properties of optimal strategies

Let us come back to an investor's maximization problem. We learned from the previous section that there are no free lunches. Hence it is sensible now to ask whether there exists a solution to an investor's dynamic optimization program. For ease of exposition, we assume that the investor seeks to maximize his expected utility of time one wealth and that his utility function is of the constant relative risk aversion type: $u(w) = \frac{1}{1-b} w^{1-b}$, with $b \neq 1$. Formally stated, the problem the investor tries to solve is:

$$\begin{aligned} & \max_{(\alpha, \theta) \in H} \frac{1}{1-b} E[w^{1-b}] \\ \text{s.t. } & \alpha(0)B(0) + \theta(0)S(0) = K_0 \\ & \alpha(1)B(1) + \theta(1)S(1) = w \in L_+^2(P), \end{aligned} \tag{3.1}$$

where K_0 is the investor's initial wealth at time zero, and where $L_+^2(P)$ is the collection of all the nonnegative elements of $L^2(P)$.

Merton [1971] has solved this problem by way of the stochastic dynamic programming. To show that there exists a solution to a problem like (3.1) using dynamic programming, there are two approaches. The first is through some existence theorems in the theory of stochastic control. Those existence theorems often require an admissible control to take its values in a compact set, but here that is not satisfactory. If we are modeling frictionless markets, any compactness assumption on the values of controls or strategies is arbitrary. Moreover, most of the treatments of stochastic control theory are extremely complicated, e.g., Krylov [1980]. More comprehensible treatments such as Fleming and Rishel [1975], do not consider cases where the controls affect the diffusion term of the controlled processes. This, unfortunately, rules out the portfolio problem under consideration. (The control referred to above is a strategy in our context.)

The second approach is through construction: construct a control or a strategy and show that it satisfies the Bellman's equation. Merton's solution to (3.1) uses the second approach. In general, however, it is difficult to construct a solution.

Using the insights from the martingale approach to the contingent claims pricing theory, we will demonstrate a technique that proves the existence of a solution to (3.1) when the space of admissible controls is a linear space. This technique is easier to understand than the theory of stochastic control and the properties of a solution can be easily characterized. Kreps [1979] was the first to recognize the possibility of this new technique.

We know from the previous sections that any element x of $L^2(P) \cap L^2(Q)$ is achievable and has a price $E^*[xe^{-r}]$. Thus the investor can be viewed as facing the static problem of choosing an element of $L_+^2(P) \cap L^2(Q)$ subject to his budget constraint. We shall consider a slightly different problem that is easier to solve.

Consider the following static maximization problem:

$$\begin{aligned} \max_{w \in L_+^2(P)} & \frac{1}{1-b} E[w^{1-b}] \\ \text{s.t. } & E^*[we^{-r}] = K_0. \end{aligned} \quad (3.2)$$

Noting that $E^*[we^{-r}] = E[we^{-r}\xi]$, we can use the Lagrangian method to get the unique solution to (3.2):

$$\hat{w} = K_0 \exp \left\{ r - \frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2 \frac{1}{b^2} + \frac{1}{b} \left(\frac{\mu - r}{\sigma} \right) z^*(1) \right\},$$

a log-normally distributed random variable under Q . Thus \hat{w} has a finite second moment under Q . That is, $\hat{w} \in L_+^2(P) \cap L^2(Q)$ and, by the martingale representative theorem, is achievable through some strategy $(\hat{\alpha}, \hat{\theta}) \in H$.

We claim that $(\hat{\alpha}, \hat{\theta})$ is a solution to (3.1). If this is not the case, there must exist a $(\alpha, \theta) \in H$ that satisfies the constraints of (3.1) such that

$$\frac{1}{1-b} E[w^{1-b}] > \frac{1}{1-b} E[\hat{w}^{1-b}],$$

where $w = \alpha(1)B(1) + \theta(1)S(1)$. We know $w \in L^2(P)$ and $E^*[we^{-r}] = K_0$, and is thus feasible in (3.2). This contradicts the fact that \hat{w} is a solution to (3.2). In fact, given that H is a linear space and that the utility function is strictly concave, $(\hat{\alpha}, \hat{\theta})$ is the unique solution to (3.1).

Using a static maximization method together with the martingale representation theorem we have demonstrated the existence of a solution to (3.1). The space of admissible controls H is a linear space, embodying the notion of frictionless markets. Here we used a specific utility function, but the idea can be applied to a large class of utility functions and a very general stochastic environment (see Cox and Huang [1986a] and Pliska [1986]).

As in option pricing theory, we can readily compute the optimal strategy. Let $f(t)$ be the value of \hat{w} at time t . We know

$$\begin{aligned} f(t)e^{-rt} &= E^*[\hat{w}e^{-r} | \mathcal{F}_t] \\ &= K_0 \exp \left\{ -\frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2 \frac{1}{b^2} + \frac{1}{b} \left(\frac{\mu - r}{\sigma} \right) z^*(t) \right\} \\ &= K_0 \left\{ -\frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2 \frac{1}{b^2} + \frac{1}{b} \frac{(\mu - r)}{\sigma^2} \ln S^*(t) + \frac{1}{2b} (\mu - r)t \right\}. \end{aligned}$$

Thus we can write $f(t) = f(S(t), t)$. It is easily checked that we can apply Itô's lemma to f . Since $f(t)e^{-rt}$ is a martingale having a finite second moment under Q , Itô's lemma implies that

$$f(S(t), t)e^{-rt} = f(0) + \int_0^t \frac{f(S(s), s)e^{-rs}}{S^*(s)} \frac{1}{b} \frac{\mu - r}{\sigma^2} dS^*(s).$$

The same arguments as in the option pricing part show that

$$\begin{aligned} \hat{\theta}(t) &= \frac{f(S(t), t)e^{-rt}}{S^*(t)} \frac{1}{b} \frac{\mu - r}{\sigma^2} \\ &= \frac{f(S(t), t)}{S(t)} \frac{1}{b} \frac{\mu - r}{\sigma^2}. \end{aligned}$$

The proportion of the investor's wealth invested in the risky asset is a constant:

$$\frac{\hat{\theta}(t)S(t)}{f(S(t), t)} = \frac{1}{b} \frac{\mu - r}{\sigma^2}.$$

By applying Itô's lemma to $f(S(t), t)$, we find that $\hat{\theta}(t)$ is equal to $f_S(S(t), t)$. Then $\hat{\alpha}(t)$ is determined through (2.12). This technique for characterizing optimal portfolio policies can be applied more generally. It is especially effective in situations in which utility is derived from intermediate consumption as well as terminal wealth. The natural nonnegativity constraint on consumption and terminal wealth can be handled with no added difficulty. For example, in the maximization of the terminal wealth case, the solution for the constrained problem can be decomposed into two parts.

The first part is an unconstrained solution for a fraction of the initial wealth and the second part is a put option written on the first part with a zero exercise price. The fraction of the initial wealth in the first part is determined such that the value of the put option in the second part exhausts the remaining initial wealth. Readers are referred to Cox and Huang [1986b] for details.

4. Applications to Contingent Claim Pricing

As is evident from the previous sections, option pricing theory can potentially be applied to any security whose future payoffs are contractually related to the value of some other security or group of securities. Option pricing theory thus leads directly to a general theory of contingent claim pricing relevant for a wide variety of financial instruments. This theory has been especially useful for practitioners because it not only provides a standard of value but also provides a production technology for duplicating the payoffs of any contingent claim by using an appropriate dynamic strategy. Consequently, it provides a constructive technique for exploiting mispricing opportunities and for hedging the risk associated with holding positions in contingent claims.

A particularly important application of contingent claim pricing theory has been in the relative valuation of corporate securities. Here the theory links the value of a single corporate security to the total value of all of the firm's outstanding securities. For example, consider a firm which has outstanding common stock and a single issue of zero coupon bonds with a promised payment B and a maturity date T . If the value of the firm on the maturity date is greater than B , the bondholders will be paid in full; otherwise, the firm will be in default and the ownership of the firm will pass to the bondholders. Thus, in a perfect and frictionless market, the value of the bonds at date T will be the maximum of B and the value of the firm. This is exactly the same as the value at date T of a portfolio containing a long position in a default-free zero coupon bond with promised payment B and maturity date T and a short position in one European put option on the value of the firm with exercise price B and expiration date T . Since the bonds and the portfolio have the same future value in all circumstances, they must have the same current value. Consequently, the proper discount to the corporate bonds for the possibility of default is exactly equal to the value of the put option. Among the early work in this area are articles by Merton [1974] and Black and Cox [1976] on ordinary bonds and Ingersoll [1977] and Brennan and Schwartz [1977b] on convertible bonds.

Some especially interesting recent work concerns the game-theoretic issues associated with corporate securities whose owners have certain discretionary conversion rights, such as warrants. Warrants differ from ordinary options in several important ways. When an option is exercised, the number of shares outstanding of the underlying stock remains unchanged and the exercise price is transferred to the individual who sold the option. As a consequence, setting aside informational issues, the exercise of any one option has no effect on the value of any other outstanding option. Therefore, all options with identical terms will optimally be exercised at the same time. By analogy,

warrant issues have traditionally been valued under the assumption that all of the warrants would be exercised simultaneously. However, when a warrant is exercised, new shares are issued and the exercise price becomes part of the assets of the firm. By employing option pricing methodology, Emanuel [1983] showed that the differences between options and warrants imply that simultaneous exercise may no longer be optimal for a warrant issue owned by a single agent. The exercise of any one warrant influences the value of the remaining ones, and a monopolist can use this situation to his advantage by in effect sacrificing some warrants for the good of the others. Competing owners of individual warrants would not have this opportunity, so the distribution of ownership may affect both the value of the warrants and the optimal pattern of their exercise. In subsequent work, Constantinides [1984] found that the total value of a warrant issue held by competing individuals is in some situations the same as that derived under the assumption of simultaneous exercise. Surprisingly, this is true even though the competing individuals may not exercise their warrants simultaneously. Constantinides and Rosenthal [1984] explicitly modeled the exercise of a warrant issue held by competing individuals as a noncooperative game and provided some existence results. Spatt and Sterbenz [1986] examined the interaction between optimal warrant exercise strategies and the firm's capital structure, dividend, and reinvestment policies. Among other things, they showed that the firm can follow policies which will eliminate any advantage to sequential exercise strategies. These papers are part of a growing body of literature on strategic issues in the valuation of corporate securities, and this will undoubtedly be an active area of research in the future.

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